

AUTOMORPHISM INVARIANCE AND IDENTITIES

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If an outer (multilinear) commutator identity holds in a large subgroup of a group, then it holds also in a large characteristic subgroup. Similar assertions are valid for algebras and their ideals or subspaces. Varying the meaning of the word “large”, we obtain many interesting facts. These results cannot be extended to arbitrary (non-multilinear) identities. As an application, we give a sharp estimate for the ‘virtual derived length’ of (virtually solvable)-by-(virtually solvable) groups.

0. Introduction

In [KhM07a], it was shown that each group virtually satisfying an outer commutator identity contains a finite-index characteristic subgroup satisfying this identity. A similar result was obtained in [KhM08] for ideals in arbitrary algebras (with codimension taking the role of index). Another similar result can be found in [KhM07b]: if a finite p -group G contains a normal class t nilpotent subgroup N , then it also contains a characteristic class t nilpotent subgroup H whose co-rank is bounded by a function of the co-rank of N and the number t . The paper [KIM09] contains a new much shorter proof of the theorem about index and a better estimate for the index of the characteristic subgroup.

In this note, we extend and generalize the new proof in [KIM09] to a wide class of algebraic systems. In this generalization the property of having finite index is replaced by a certain abstract property of “smallness”, and index is replaced by a certain abstract “codimension”. This general theorem includes all the aforementioned results as special cases. Moreover, there are many other new applications, which include, for example, the case of finite p -groups with subgroups of bounded co-rank satisfying arbitrary outer commutator identity (rather than just nilpotency identity as in [KhM07b]).

On the other hand, we show that the original proofs in [KhM07a], [KhM07b], and [KhM08] yield a bit more. Namely, each group contains only finitely many finite-index subgroups which are maximal (by inclusion) among all normal subgroups satisfying a given outer commutator identity. In particular, this implies that each finite-index subgroup which is maximal among all normal subgroups satisfying a given outer commutator identity contains a finite-index characteristic subgroup. Similar stronger results are valid for algebras over fields.

To complete the picture, we mention earlier known results on this subject. Let G be a group and let N be its finite-index subgroup. Then

- N contains a normal in G subgroup of finite index (dividing $|G : N|$);
- if G is finitely generated, then N contains a fully invariant (and even verbal) in G subgroup of finite index;
- if N is abelian, then G contains a characteristic abelian subgroup of finite index.

These facts are well known and can be found in textbooks on group theory (see, e.g., [KaM82]). Note also that in [BeK03] it was proved that the existence of a solvable finite-index subgroup of derived length t implies the existence of a characteristic solvable finite-index subgroup of derived length $\leq t^2$.

As an application of the obtained results, in Section 5 we obtain a sharp estimate for the ‘virtual derived length’ of extensions of virtually solvable groups by virtually solvable groups. This answers a question of J. Button.

In Section 6, we consider the periodicity law $x^p = 1$. We show that the theorem about finite-index characteristic subgroup does not hold for this identity (if p is a large prime).

1. The results

Theorem 1 [KhM07a], [KIM09]. *If a group G contains a normal finite-index subgroup N satisfying an outer commutator identity $w(x_1, \dots, x_t) = 1$, then G contains a characteristic and even invariant under all surjective endomorphisms subgroup H satisfying the same identity and such that $\log_2 |G : H| \leq f^{t-1}(\log_2 |G : N|)$.*

Henceforth, $f^k(x)$ means the k -th iteration of the function $f(x) = x(x+1)$. An *outer* (or *multilinear*) *commutator identity* is an identity of the form $[\dots [x_1, \dots, x_t] \dots] = 1$ with some meaningful arrangement of brackets, where all letters x_1, \dots, x_t are different. Examples of such identities are solvability, nilpotency, centre-by-metabelianity, etc. A formal definition looks as follows. Let $F(x_1, x_2, \dots)$ be a free group of countable rank. An *outer commutator of weight 1* is just a letter x_i . An *outer commutator of weight $t > 1$* is a word of the form $w(x_1, \dots, x_t) = [u(x_1, \dots, x_r), v(x_{r+1}, \dots, x_t)]$, where u and v are outer commutators of weights r and $t-r$, respectively. An *outer commutator identity* is an identity of the form $w = 1$, where w is an outer commutator.

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Remark 1. The condition that the subgroup N be normal is not essential. It is well known that any finite-index subgroup N contains a normal finite-index subgroup \tilde{N} such that $|G:\tilde{N}|$ does not exceed (and even divides) $|G:N|!$ (see, e.g., [KaM82]). Therefore, Theorem 1 remains valid for non-normal subgroups N , but with worse estimate $\log_2 |G:H| \leq f^{t-1}(\log_2 |G:N|!)$.

Remark 2. Theorem 1' (see below) implies that, under the conditions of Theorem 1, a characteristic (and even invariant under all surjective endomorphisms) finite-index subgroup can be found inside any finite-index subgroup which is maximal (by inclusion) among all normal subgroups satisfying the outer commutator identity.

Remark 3. Theorem 4 (see below), which is a generalization of Theorem 1, implies that the group G/H lies in the variety generated by G/N (and even in the formation generated by this group).

Theorem 2 (cf. [KhM08]). *Let G be an algebra (possibly, non-associative) over a field. If G contains a finite-codimensional subspace N satisfying a multilinear identity $w(x_1, \dots, x_t) = 0$, then G contains a subspace H satisfying the same identity, invariant under all surjective endomorphisms, and such that $\text{codim } H \leq f^{t-1}(\text{codim } N)$. This subspace H is left, right, or two-sided ideal if the subspace N is left, right, or two-sided ideal, respectively.*

Remark. Theorem 2' (see below) implies that, under the conditions of Theorem 2, a finite-codimensional subspace invariant under all surjective endomorphisms can be found inside any finite-codimensional subspace which is maximal (by inclusion) among all subspaces satisfying the multilinear identity. A similar fact is valid for ideals (left, right, and two-sided).

Theorem 3 (cf. [KhM07b]). *If a finite p -group G contains a normal subgroup N satisfying an outer commutator identity $w(x_1, \dots, x_t) = 1$, then G contains a characteristic subgroup H satisfying the same identity and such that $\text{rank } G/H \leq f^{t-1}(\text{rank } G/N)$.*

Here, $\text{rank } G$ is the minimal positive integer n such that any finitely generated subgroup of G is generated by at most n elements.

Theorem 4. *If a group G contains a normal subgroup N satisfying an outer commutator identity $w(x_1, \dots, x_t) = 1$ and G/N has a smallness property \mathcal{P} , then G contains a characteristic and even invariant under all surjective endomorphisms subgroup H , satisfying the same identity and such that G/H has the property \mathcal{P} .*

A *smallness property* in Theorem 4 is any abstract group property \mathcal{P} satisfying the following conditions:

- 1) a quotient group of a group with property \mathcal{P} also has this property;
- 2) a subdirect product of two groups with property \mathcal{P} also has this property;
- 3) each group with property \mathcal{P} satisfies the maximality (ACC) condition for normal subgroups.

Examples of such properties are the maximality condition, the maximality condition for normal subgroups, polycyclicity, finiteness, etc.

Theorem 1 was proved in [KhM07a], but with a worse estimate for the index. A simpler proof and the estimate presented above was obtained in [KIM09]. Theorem 2 was proved in [KhM08] with a worse estimate for the codimension. An important particular case of Theorem 3 corresponding to the nilpotency identity was proved in [KhM07b]. Theorem 4 is new. All these theorems turn out to be special cases of a general fact concerning multi-operator groups.

The methods developed in [KhM07a], [KhM07b], and [KhM08] allows us to prove assertions stronger than Theorems 1 and 2, but with worse estimates.

Theorem 1'. *Let $w(x_1, \dots, x_t)$ be an outer commutator. Then, in any group, the number of finite-index subgroups which are maximal (by inclusion) among all normal subgroups satisfying the identity $w(x_1, \dots, x_t) = 1$ is finite. Moreover, the number of such subgroups of index $\leq n$ does not exceed*

$$2^{F^{t-1}(n)}, \quad \text{where } F^k(x) \text{ is the } k\text{-th iteration of the function } F(x) = xn^{2^x}.$$

Remark. This theorem consists of two independent assertions. On the one hand, the number of subgroups of index $\leq n$ which are maximal among all normal subgroups with the identity $w(x_1, \dots, x_t) = 1$ is bounded by an explicit function of n and t . This function grows very fast, but on the other hand, the total number of finite-index subgroups which are maximal among all normal subgroups with given identity is finite. The following theorem shows that a similar statement is valid for subspaces (or ideals) in algebras.

Theorem 2'. *Let $w(x_1, \dots, x_t)$ be a multilinear element of the free (non-associative) algebra over a field F . Then, in any algebra over F , the intersection of all finite-codimensional ideals which are maximal (by inclusion) among all ideals satisfying the identity $w(x_1, \dots, x_t) = 0$ has finite codimension. Moreover, the intersection of such ideals of codimension $\leq n$ has codimension not larger than some number depending only on n and t . Here, the word "ideal" means left, right, two-sided ideal, or simply subspace (zero-side ideal).*

Theorem 1 makes it possible to give the most exact answer to a question of J. O. Button ([But08], Problem 3) on extensions of virtually solvable groups by virtually solvable groups.

Theorem 5. Any extension of a virtually solvable of derived length s group by a virtually solvable of derived length t group is virtually solvable of derived length $\leq t + s + 1$.

In section 5, we prove this theorem and give a simple example showing that the obtained estimate is best possible.

The following assertion shows that neither Theorem 1 nor other versions of Theorem 4 can be extended to arbitrary identities.

Theorem 6. For any sufficiently large prime p , there exists a group G of exponent p^2 with a finite-index subgroup of exponent p , but without characteristic finite-index subgroups of exponent p . Moreover, any quotient of G by a characteristic subgroup of exponent p satisfies neither the maximality condition for normal subgroups nor the minimality condition for normal subgroups and, therefore, has no smallness properties.

2. Multi-operator groups

Recall that, according to [Kur62], an Ω -group is a group $(G, +)$ (not necessarily commutative) with a family of operations Ω . Each operation $f \in \Omega$ is a mapping $f : G^{n_f} \rightarrow G$ from a finite Cartesian power of G to G such that $f(0, \dots, 0) = 0$.

Examples.

1. Any group can be considered as an Ω -group with empty set of operations Ω .
2. A ring is an Ω -group with commutative addition and the set of operations Ω consisting of one binary operation (multiplication), satisfying the distributivity law.
3. An algebra over a fixed field F can be considered as an Ω -group with commutative addition and the set of operations Ω consisting of one binary operation (multiplication) and $|F|$ unary operations (multiplication by scalars) satisfying the well-known laws.

Let \mathcal{V} be a variety of Ω -groups and let $w(x_1, \dots, x_t)$ be an element of the free (in the variety \mathcal{V}) algebra $F_{\mathcal{V}}(x_1, \dots, x_t)$. For additive normal subgroups A_1, \dots, A_t of an Ω -group $G \in \mathcal{V}$, we define $w(A_1, \dots, A_t)$ as the normal subgroup generated by all elements of the form $w(a_1, \dots, a_t)$, where $a_i \in A_i$.

An element $w(x_1, \dots, x_t) \in F_{\mathcal{V}}(x_1, \dots, x_t)$ is called *multilinear* if

$$w(A_1, \dots, A_i + A'_i, \dots, A_t) = w(A_1, \dots, A_i, \dots, A_t) + w(A_1, \dots, A'_i, \dots, A_t)$$

for all $i = 1, \dots, t$ and any normal subgroups $A_1, \dots, A_i, A'_i, \dots, A_t$ of any Ω -group of the variety \mathcal{V} .

The outer commutators are multilinear elements of the absolutely free group. The multilinear (in usual sense) expressions are multilinear elements of the free algebra over a field.

Let \mathcal{C} be a class of normal subgroups of an Ω -group G such that

- 1) \mathcal{C} is closed with respect to the images under surjective endomorphisms of the Ω -group G , finite sums and finite intersections;
- 2) any subfamily $\mathcal{N} \subseteq \mathcal{C}$ of the class \mathcal{C} contains a finite subfamily $\mathcal{F} \subseteq \mathcal{N}$ such that

$$\sum_{N \in \mathcal{N}} N = \sum_{N \in \mathcal{F}} N.$$

In this case, we say that \mathcal{C} is a *class of large normal subgroups*. A function $\overline{\text{codim}} : \mathcal{C} \rightarrow \mathbb{R}$ is called a (*generalized*) *codimension* if it has the following properties:

- 0) $\overline{\text{codim}} N_1 \leq \overline{\text{codim}} N_2$ if $N_1 \supseteq N_2$;
- 1) $\overline{\text{codim}} \varphi(N) \leq \overline{\text{codim}} N$ for each subgroup $N \in \mathcal{C}$ and each surjective endomorphism of the Ω -group G ;
- 2) $\overline{\text{codim}} (N_1 \cap N_2) \leq \overline{\text{codim}} N_1 + \overline{\text{codim}} N_2$ for all subgroups $N_1, N_2 \in \mathcal{C}$;
- 3) in any family \mathcal{N} of subgroups from the class \mathcal{C} , there exist $r \leq \max_{N \in \mathcal{N}} \overline{\text{codim}} N + 1$ subgroups N_1, \dots, N_r such that

$$\sum_{N \in \mathcal{N}} N = \sum_{i=1}^r N_i.$$

If G is an algebra over a field and the class \mathcal{C} consists of all subspaces or all ideals (one-sided or two-sided) of finite codimension, then the usual codimension can be regarded as a generalized codimension.

If G is a group and the class \mathcal{C} consists of all normal finite-index subgroups, then, as a generalized codimension, we can take the binary logarithm of the index. If the class \mathcal{C} consists of all normal subgroups whose index is a power of a fixed prime p , then, as a codimension of a subgroup N , we can take the rank of the quotient G/N (Property 3 holds by virtue of the Burnside basis theorem).

The class of all normal subgroups such that the corresponding quotient groups satisfy a fixed smallness property \mathcal{P} can also be regarded as a class of large subgroups. However, no codimension is defined in this case.

3. Main theorem

In the proof of the main theorem, we follow the argument in [KIM09] generalizing it to multi-operator groups.

Lemma 1. Suppose that $w(x_1, \dots, x_t)$ is a multilinear element of the free Ω -group of some variety \mathcal{V} , m is a positive integer, $G \in \mathcal{V}$ is an Ω -group, and \mathcal{N} is a finite family of its normal subgroups such that

$$w(\underbrace{N, N, \dots, N}_{m \text{ times}}, G, G, \dots, G) = 0 \quad \text{for all } N \in \mathcal{N}.$$

Then

$$w(\underbrace{\hat{N}, \hat{N}, \dots, \hat{N}}_{m-1 \text{ times}}, \hat{G}, \hat{G}, \dots, \hat{G}) = 0, \quad \text{where } \hat{N} = \bigcap_{N \in \mathcal{N}} N \text{ and } \hat{G} = \sum_{N \in \mathcal{N}} N.$$

Proof.

$$w(\underbrace{\hat{N}, \hat{N}, \dots, \hat{N}}_{m-1 \text{ times}}, \hat{G}, \hat{G}, \dots, \hat{G}) = w(\underbrace{\hat{N}, \hat{N}, \dots, \hat{N}}_{m-1 \text{ times}}, \sum_{N \in \mathcal{N}} N, \hat{G}, \dots, \hat{G}) = \sum_{N \in \mathcal{N}} w(\underbrace{\hat{N}, \hat{N}, \dots, \hat{N}}_{m-1 \text{ times}}, N, \hat{G}, \dots, \hat{G}).$$

But $\hat{N} \subseteq N$ and $\hat{G} \subseteq G$; therefore each term of the last sum is contained in the normal subgroup

$$w(\underbrace{N, N, \dots, N}_{m \text{ times}}, G, G, \dots, G), \quad \text{which is trivial by assumption.}$$

As a corollary, we obtain the main theorem.

Main theorem. Suppose that G is an Ω -group belonging to a variety \mathcal{V} , \mathcal{C} is a class of its large normal subgroups, $w(x_1, \dots, x_t) \in F_{\mathcal{V}}(x_1, \dots, x_t)$ is a multilinear element, $N \in \mathcal{C}$, and $w(N, \dots, N) = 0$. Then G contains an invariant under all surjective endomorphisms normal subgroup $H \in \mathcal{C}$ satisfying the same identity $w(H, \dots, H) = 0$. In addition, if $\overline{\text{codim}}: \mathcal{C} \rightarrow \mathbb{R}$ is a generalized codimension, then

$$\overline{\text{codim}} H \leq f^{t-1}(\overline{\text{codim}} N),$$

where $f^k(x)$ is the k -th iteration of the function $f(x) = x(x+1)$.

Proof. Let $\text{Ends } G$ be the semigroup of all surjective endomorphisms of the Ω -group G . Consider the normal subgroup $G_1 = \sum_{\varphi \in \text{Ends } G} \varphi(N)$. This subgroup is invariant under all surjective endomorphisms, is large (i.e. belongs to \mathcal{C}), and $\overline{\text{codim}} G_1 \leq \overline{\text{codim}} N$ (if $\overline{\text{codim}}$ is defined). Clearly, G_1 is the sum of a finite number of the images of N (because N is large) and this finite number does not exceed $\overline{\text{codim}} N + 1$ (by the definition of the codimension). Thus,

$$G_1 = \sum_{k=0}^{p_1} \varphi'_k(N), \quad \text{where } \varphi'_k \in \text{Ends } G \text{ and } p_1 \leq l_0 \stackrel{\text{def}}{=} \overline{\text{codim}} N.$$

Now, consider the normal subgroup $N_1 = \bigcap_{k=0}^{p_1} \varphi'_k(N)$. Clearly, this subgroup is large too. By Properties 1) and 2) of the codimension $\overline{\text{codim}}$, we have

$$l_1 \stackrel{\text{def}}{=} \overline{\text{codim}} N_1 \leq (p_1 + 1) \overline{\text{codim}} N = (p_1 + 1) l_0 \leq (l_0 + 1) l_0 = f(l_0).$$

According to Lemma 1,

$$w(N_1, \dots, N_1, G_1) = 0.$$

Similarly, we construct the large normal subgroups

$$G_2 = \sum_{\varphi \in \text{Ends } G} \varphi(N_1) = \sum_{k=0}^{p_2} \varphi''_k(N_1) \quad \text{and} \quad N_2 = \bigcap_{k=0}^{p_2} \varphi''_k(N_1), \quad \text{where } \varphi''_k \in \text{Ends } G \text{ and } p_2 \leq \overline{\text{codim}} N_1 = l_1 \leq f(l_0).$$

Clearly, G_2 is invariant under all surjective endomorphisms of the Ω -group G ,

$$\overline{\text{codim}} G_2 \leq \overline{\text{codim}} N_1 = l_1 \leq f(l_0), \quad \text{and} \quad l_2 \stackrel{\text{def}}{=} \overline{\text{codim}} N_2 \leq (p_2 + 1) \overline{\text{codim}} N_1 = (p_2 + 1) l_1 \leq f(l_1) \leq f(f(l_0)).$$

According to Lemma 1,

$$w(N_2, \dots, N_2, G_2, G_2) = 0.$$

Continuing in the same manner, at the t -th step, we obtain in G an invariant under all surjective endomorphisms large normal subgroup

$$G_t = \sum_{\varphi \in \text{Ends } G} \varphi(N_{t-1}) = \sum_{k=0}^{p_t} \varphi_k^{(t)}(N_{t-1}), \quad \text{where } \varphi_k^{(t)} \in \text{Ends } G.$$

For this subgroup, we have

$$w(G_t, \dots, G_t) = 0 \quad \text{and} \quad \overline{\text{codim}} G_t \leq \overline{\text{codim}} N_{t-1} = l_{t-1} \leq f(l_{t-2}) \leq f(f(l_{t-3})) \leq \dots \leq f^{t-1}(l_0).$$

Thus, the subgroup $H = G_t$ is as required and the theorem is proved.

Theorems 1 – 4 are special cases of the main theorem:

Theorem 1: $\mathcal{V} = \{\text{Groups}\}$, $\mathcal{C} = \{\text{Normal subgroups of finite index}\}$, $\overline{\text{codim}} N = \log_2 |G:N|$;

Theorem 2: $\mathcal{V} = \{\text{Algebras}\}$, $\mathcal{C} = \left\{ \begin{array}{l} \text{Subspaces or ideals} \\ \text{of finite codimension} \end{array} \right\}$, $\overline{\text{codim}} = \text{codim}$;

Theorem 3: $\mathcal{V} = \{\text{Groups}\}$, $\mathcal{C} = \{N \triangleleft G ; G/N \text{ is a } p\text{-group}\}$, $\overline{\text{codim}} N = \text{rank } G/N$;

Theorem 4: $\mathcal{V} = \{\text{Groups}\}$, $\mathcal{C} = \{N \triangleleft G ; G/N \text{ has the property } \mathcal{P}\}$, $\overline{\text{codim}}$ is not defined.

4. Proof of Theorems 1' and 2'

Theorem 1' follows from the following proposition.*)

Proposition 1. Suppose that a group G contains sufficiently many normal subgroups N_1, \dots, N_m of index n satisfying an outer commutator identity $w(x_1, \dots, x_t) = 1$ (where m is a sufficiently large number depending only on n and t). If

$$\bigcap_{j \neq k} N_j \neq \bigcap_{j \neq k} N_j \quad \text{for all } k = 1, \dots, m,$$

then the group G has a normal subgroup X satisfying the same identity and strictly containing one of the subgroups N_j (and, hence, the index of X is strictly less than n).

Proof. This proposition was, actually, proved in [KhM07a]. The subgroup X constructed in the proof of Proposition 1 of [KhM07a] contains one of the subgroups N_j .

Theorem 2' similarly follows from the following proposition in [KhM08].*)

Proposition 2 ([KhM08], Proposition 3). Let N_1, \dots, N_m be ideals of an algebra A over a field K and let $f(x_1, \dots, x_c) \in K \langle x_1, \dots, x_c \rangle$ be a multilinear polynomial. Suppose that

(a) each ideal N_i satisfies the identity $f = 0$ and $\dim A/N_i \leq r$;

(b) $\bigcap_{j \neq k} N_j \neq \bigcap_{j \neq k} N_j$ for all $k = 1, \dots, m$.

If $m \geq s(r, c)$ for some (r, c) -bounded number $s(r, c)$, then there exists $k \in \{1, \dots, m\}$ such that the ideal $N_k + \bigcap_{j \neq k} N_j$ satisfies the identity $f = 0$.

For the reader's convenience, we give an independent and simpler proof of Theorem 1'.

Proof of Theorem 1'. Let \mathcal{N} be a set of finite-index subgroups of G that are maximal by inclusion among all normal subgroups satisfying the outer commutator identity $w(x_1, \dots, x_t) = 1$. We must prove that \mathcal{N} is finite.

If the family \mathcal{N} is empty, then we have nothing to prove. Otherwise, consider a subgroup $G_0 \in \mathcal{N}$. This subgroup satisfies the identity

$$w_\sigma(G_0, \dots, G_0) = 1 \quad \text{for all } \sigma \in S_t. \quad (0)$$

Henceforth, $w_\sigma(x_1, \dots, x_t)$ denotes $w(x_{\sigma(1)}, \dots, x_{\sigma(t)})$, where σ is a permutation of degree t .

*) To be more precise, Proposition 1 implies the assertion of Theorem 1' on subgroups of bounded index. The finiteness of the total number of finite-index maximal normal subgroups with given identity follows from the proof of this proposition. The relations between Proposition 2 and Theorem 2' are similar.

The subgroup G_0 has finite index. Therefore, the family of subgroups $\{NG_0 \mid N \in \mathcal{N}\}$ is finite and coincides with the family $\{NG_0 \mid N \in \mathcal{N}_1\}$, where \mathcal{N}_1 is a finite subfamily of the family \mathcal{N} . The subgroup

$$G_1 = G_0 \cap \bigcap_{N \in \mathcal{N}_1} N$$

has finite index and satisfies the equality

$$w_\sigma(G_1, \dots, G_1, NG_0) = 1 \quad \text{for all } \sigma \in S_t \quad \text{and for all } N \in \mathcal{N}. \quad (1)$$

Indeed, by the choice of the family \mathcal{N}_1 , each product NG_0 , where $N \in \mathcal{N}$, coincides with a product $N_1 G_0$ for some group $N_1 \in \mathcal{N}_1$ and $N_1 \supseteq G_1 \subseteq G_0$. Therefore,

$$\begin{aligned} w_\sigma(G_1, \dots, G_1, NG_0) &= w_\sigma(G_1, \dots, G_1, N_1 G_0) = w_\sigma(G_1, \dots, G_1, N_1) w_\sigma(G_1, \dots, G_1, G_0) \subseteq \\ &\subseteq w_\sigma(N_1, \dots, N_1, N_1) w_\sigma(G_0, \dots, G_0, G_0) = 1. \end{aligned}$$

The subgroup G_1 has finite index. Therefore, the family of subgroups $\{NG_1 \mid N \in \mathcal{N}\}$ is finite and coincides with the family $\{NG_1 \mid N \in \mathcal{N}_2\}$, where \mathcal{N}_2 is a finite subfamily of the family \mathcal{N} . The subgroup

$$G_2 = G_1 \cap \bigcap_{N \in \mathcal{N}_2} N$$

has finite index and satisfies the equality

$$w_\sigma(G_2, \dots, G_2, NG_1, NG_1) = 1 \quad \text{for all } \sigma \in S_t \quad \text{and for all } N \in \mathcal{N}. \quad (2)$$

Indeed, by the choice of the family \mathcal{N}_2 , each product NG_1 , where $N \in \mathcal{N}$, coincides with a product $N_2 G_1$ for some group $N_2 \in \mathcal{N}_2$ and $N_2 \supseteq G_2 \subseteq G_1 \subseteq G_0$. Therefore,

$$\begin{aligned} w_\sigma(G_2, \dots, G_2, NG_1, NG_1) &= w_\sigma(G_2, \dots, G_2, N_2 G_1, N_2 G_1) = \\ &= w_\sigma(G_2, \dots, G_2, N_2, N_2) w_\sigma(G_2, \dots, G_2, N_2, G_1) w_\sigma(G_2, \dots, G_2, G_1, N_2) w_\sigma(G_2, \dots, G_2, G_1, G_1) \subseteq \\ &\subseteq w_\sigma(N_2, \dots, N_2, N_2, N_2) w_\sigma(G_1, \dots, G_1, N_2, G_1) w_\sigma(G_1, \dots, G_1, G_1, N_2) w_\sigma(G_0, \dots, G_0, G_0, G_0). \end{aligned}$$

The first factor of the last product is trivial, because the group N_2 satisfies the identity $w = 1$. The second and the third factors are trivial by (1). The fourth factor is trivial by (0).

Continuing in the same manner, we finally obtain a finite index subgroup G_{t-1} such that

$$w_\sigma(NG_{t-1}, \dots, NG_{t-1}) = 1 \quad \text{for all } \sigma \in S_t \quad \text{and for all } N \in \mathcal{N}. \quad (t)$$

By virtue of the maximality of all these subgroups N , this means that $G_{t-1} \subseteq N$ for all $N \in \mathcal{N}$, i.e. $G_{t-1} \subseteq \bigcap_{N \in \mathcal{N}} N$ and, therefore, this intersection has finite index. The finiteness of this index implies the finiteness of the family \mathcal{N} , as required.

To obtain an estimate, it is sufficient to note that, if all subgroups from the family \mathcal{N} have index not larger than n , then

$$|G : G_k| \leq |G : G_{k-1}| n^{|\mathcal{N}_k|} \quad \text{and} \quad |\mathcal{N}_k| \leq 2^{|G : G_{k-1}|} \quad (\text{this is a very rough estimate}).$$

Therefore,

$$|G : G_k| \leq |G : G_{k-1}| \cdot n^{2^{|G : G_{k-1}|}}, \quad \text{i.e.} \quad |G : G_{t-1}| \leq F^{t-1}(n) \quad \text{and} \quad |\mathcal{N}| \leq 2^{F^{t-1}(n)},$$

where $F^k(x)$ is the k th iteration of the function $F(x) = xn^{2^x}$.

5. Meta-virtually-solvable groups

In this section, we prove Theorem 5, i.e. we obtain the best possible estimate on the ‘virtual derived length’ of a group G containing a normal virtually solvable of derived length s subgroup A such that the quotient G/A is virtually solvable of derived length t . The second “virtually” can easily be removed. Indeed, replacing the group G by its finite-index subgroup (the preimage of the solvable finite-index subgroup of the quotient G/A), we can assume that the quotient G/A is solvable of derived length t .

Next, by Theorem 1, the solvable of derived length s finite-index subgroup N of A can be assumed to be characteristic in A and, hence, normal in G . To prove Theorem 5, it remains to show that the quotient $H = G/N$ contains a finite-index subgroup which is solvable of derived length $\leq t + 1$. But H is an extension of the finite group $K = A/N$ by the solvable of derived length t group G/A .

A finite group K has only finite number of automorphisms. Therefore, the centralizer of this group in H is of finite index. Thus, passing to a finite-index subgroup, we can assume that K is contained in the centre of H . The central quotient of H has derived length t and, therefore, H itself is solvable of derived length $\leq t + 1$, as required.

It is easy to see that the estimate in Theorem 5 cannot be improved. Indeed, consider, e.g., the central product G (i.e. the direct product with amalgamated centres) of infinitely many copies of the quaternion group of order 8. This group G is an extension of its finite (central) subgroup (of order 2) by the elementary abelian 2-group of infinite rank. It is easy to verify that this group has no abelian subgroups of finite index. (To show this, one can use Theorem 1 once again: if there is an abelian finite-index subgroup, then there exists a characteristic abelian finite-index subgroup).

Thus, an extension of a finite group (i.e. a virtually trivial group, or a virtually solvable of derived length zero group) by an abelian group is not necessarily virtually abelian. This example can be modified to construct an extension of a virtually abelian group by an abelian group which is not virtually metabelian. Indeed, take a faithful complex representation $\varphi: G \rightarrow \mathbf{GL}(V)$ (e.g., the regular one) of the group G described above. The corresponding semidirect product $G_1 = V \rtimes G$ is an extension of the virtually abelian group $A = V \rtimes \{\pm 1\}$ by an (elementary) abelian group $G_1/A \simeq G/\{\pm 1\}$. Suppose that H is a finite-index subgroup of G_1 . Let us show that H cannot be metabelian. Indeed, H must contain V (because V , being a complex vector space, has no proper finite-index subgroups). The quotient $G_1/V \simeq G$ contains a finite-index subgroup H/V . Therefore, H/V is nonabelian (since G has no finite-index abelian subgroups). Therefore, the commutator subgroup $(H/V)'$ contains a nontrivial element g of order 2. Hence, the commutator subgroup H' of H contains an element $x = ug$ (for some $u \in V$) and all elements of the form

$$[x, v] = xvx^{-1}(-v) = \varphi(g)v - v, \quad \text{where } v \in V.$$

Since the representation φ is faithful, the space V contains a vector v not lying in the kernel of the operator $\varphi(g) - \text{id}$. Thus, H' contains a nonzero vector $w = \varphi(g)v - v$ such that $\varphi(g)w = v - \varphi(g)v = -w$. So, $[x, w] = -2w \neq 0$, i.e. H' is nonabelian and H is not metabelian, as required.

Examples of higher derived lengths can be constructed in similar fashion.

6. The Burnside identity

In this section, we prove Theorem 6, i.e. we present a group virtually satisfying the identity $x^p = 1$, but having no large characteristic subgroups of period p . To construct such a group G , we use the well-known technique dealing with periodic relations. We follow the book [Olsh89]. A similar construction can be implemented on the base of the book [Adyan75].

Consider an infinite alphabet $X = \{a, x_1, x_2, \dots\}$ and the free group $G(0) = F(X)$ with basis X . The groups $G(i) = \langle X \mid R_i \rangle$, where $i \geq 1$, are defined inductively as follows. Choose a set P_i of words of length i in the group $F(X)$ having the following properties:

- 1) no word in the set P_i is conjugate in $G(i-1)$ to a power of a shorter word;
- 2) different words in the set P_i are not conjugate in $G(i-1)$ to each other and to the inverses of each other;
- 3) the set P_i is maximal (by inclusion) among all sets satisfying conditions 1) and 2).

The words in the set P_i are called *periods of rank i* . Let us define a group $G(i) = \langle X \mid R_i \rangle$ by setting

$$R_0 = \emptyset \quad \text{and} \quad R_i = R_{i-1} \cup \{u^{n_u} = 1 \mid u \in P_i\} \text{ for } i \geq 1,$$

where n_u are some positive integers (depending on u).

Clearly, each nonidentity element of the group

$$G = G(\infty) = \left\langle X \mid \bigcup_{i=1}^{\infty} R_i \right\rangle$$

is conjugated to a power of a period (of some rank). It is known that, if all numbers n_u are sufficiently large and odd, then, in the group $G(\infty)$, the order of each period u is precisely n_u (see, e.g., [Olsh89], Theorem 26.4).

Choose a sufficiently large prime p and put

$$n_u = \begin{cases} p & \text{if } \varphi(u) \notin \langle a \rangle \setminus \{1\}; \\ p^2 & \text{if } \varphi(u) \in \langle a \rangle \setminus \{1\}, \end{cases}$$

where $\varphi: F(X) \rightarrow \langle a \rangle_p \times \langle x_1 \rangle_p \times \langle x_2 \rangle_p \times \dots$ is the natural homomorphism of the free group onto the elementary abelian p -group. Thus, in the group $G = G(\infty)$, the order of each period u is either p or p^2 depending on the value of $\varphi(u)$.

Lemma 2. *If the prime p is sufficiently large, then*

- 1) *the group G is periodic of exponent p^2 ;*
- 2) *the homomorphism φ induces a homomorphism (denoted by the same letter) of the group G onto the elementary abelian p -group;*
- 3) *an element g of G has order p^2 if and only if $\varphi(g) \in \langle a \rangle \setminus \{1\}$ (the orders of other nonidentity elements are p);*
- 4) *for each positive integer i and each integer k , the mapping $f_{i,k}: X \rightarrow G$ that maps letter x_i to $a^k x_i$ and fixes the other letters of the alphabet X extends to an automorphism of G .*

Proof. The first assertion is true, because each element is conjugate to a power of a period and the periods have orders p and p^2 . The second assertion follows from the form $u^p = 1$ of the defining relations.

To prove the third assertion, consider an element g of the group G . This element is conjugate to a power of a period: $g = t^{-1}u^k t$. If p divide k , then the equality $u^{p^2} = 1$ (valid for any period u) implies that the order of g is either p or 1. On the other hand, $\varphi(g) = \varphi(u^k) = 1$. So, in this case, the assertion 3) is true. If p does not divide k , then the order of g coincides with the order of the period u . On the other hand, the inclusion $\varphi(g) \in \langle a \rangle \setminus \{1\}$ is equivalent to the inclusion $\varphi(u) \in \langle a \rangle \setminus \{1\}$ and the assertion follows from the above remark on the orders of periods.

Let us prove the fourth assertion. Note that it is sufficient to show that the mappings under consideration extend to endomorphisms, because, if this is true, then the endomorphisms $f_{i,k}$ and $f_{i,-k}$ are mutually inverse.

To verify that the mapping $f_{i,k}$ can be extended to an endomorphism, it is sufficient to show that $f_{i,k}$ transforms the defining relations into valid equalities in G . Consider a defining relation $u^l = 1$, where l is either p or p^2 depending on the value of $\varphi(u)$. The mapping $f_{i,k}$ induces an automorphism of the elementary abelian p -group that fixes each element of the subgroup $\langle a \rangle$. Therefore, $\varphi(f_{i,k}(u)) \in \langle a \rangle \setminus \{1\}$ if and only if $\varphi(u) \in \langle a \rangle \setminus \{1\}$. Thus, by assertion 3), the element $f_{i,k}(u)$ has the same order l as the period u . Hence, the mapping $f_{i,k}$ transforms all the defining relations into valid equalities and Lemma 2 is proved.

Proof of Theorem 6. The group G constructed above contains the subgroup $N = \langle x_1, x_2, \dots \rangle \ker \varphi$ of index p . According to Lemma 2(3), this subgroup satisfies the identity $x^p = 1$.

Now, consider a characteristic subgroup H of G . This subgroup is invariant, in particular, under the automorphisms $f_{i,k}$. The automorphisms $f_{i,k}$ induce automorphisms of the elementary abelian p -group

$$E = \langle a \rangle_p \times \langle x_1 \rangle_p \times \langle x_2 \rangle_p \times \dots$$

Therefore, the image $\varphi(H)$ of H is a subgroup of E invariant under all automorphisms $f_{i,k}$. But any such invariant subgroup of E is either trivial or containing a . Therefore, H is either contained in $\ker \varphi$ (and, hence, the quotient G/H is infinite and even has both-side infinite chains of normal subgroups) or containing an element of order p^2 (by Lemma 2(3)) and H does not satisfy the identity $x^p = 1$. This completes the proof of Theorem 6.

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